International Journal of Theoretical Physics, Vol. 22, No. 10, 1983

A Finite-Dimensional Quark Model

Stanley P. Gudder

Department of Mathematics and Computer Science, University of Denver, Denver, Colorado, 80208

Received April 14, 1982

A description of quarks is given in terms of a finite-dimensional Hilbert space model. Color and flavor observables are defined and the corresponding motion and energy observables are constructed using the methods of finite-dimensional quantum mechanics. It is shown that a fundamental color condition implies that quarks can only combine as mesons, baryons, antibaryons, and collections of these. Baryon and meson Hamiltonians are proposed and various masses are computed.

1. INTRODUCTION

In a previous paper (Gudder, 1982) we showed that pure quark states can be represented by unit vectors in a finite-dimensional Hilbert space $V = \mathbb{C}^{72}$. This results in a finite-dimensional model for the description of quarks. The space V admits a decomposition into a tensor product of five subspaces each of which is determined by a parameter. These parameters are color, generation, charge, type, and spin. The tensor product of the generation and charge subspaces forms the flavor space.

After defining the color and flavor observables, we apply the methods of finite-dimensional quantum mechanics (Gudder and Naroditsky, 1981) to construct corresponding motion and energy observables. We define a class of vectors in the quark tensor space called color vectors. Assuming that the color vectors are the only observable physical vector states, it follows that quarks can only combine as mesons, baryons, antibaryons and collections of these. We then classify baryons and mesons according to their statistics, generation, and quark composition. Finally, we propose baryon and meson Hamiltonians and test these Hamiltonians against known particle masses.

0020-7748/83/1000-0947$03.00/0

2. FINITE-DIMENSIONAL QUANTUM MECHANICS

This section summarizes the basic principles of finite-dimensional quantum mechanics which will be needed in the sequel. For details and a more complete discussion, we refer the reader to Gudder and Naroditsky (1981). Let r be a positive integer and let $e_1,\ldots,e_r$ be the standard basis for the r-dimensional complex Hilbert space $V=\mathbb{C}^r$. We define the *finite Fourier transform* F_r: $V\to V$ as

$$(\mathsf{F}_r f)(j)=r^{-1/2}\sum_{k=1}^{r} f(k)e^{2\pi ijk/r},\qquad i=(-1)^{1/2}, j=1,\ldots,r$$

Then F_r is unitary and

$$(\mathsf{F}_r^* f)(j)=r^{-1/2}\sum_{k=1}^{r} f(k)e^{-2\pi ijk/r},\qquad j=1,\ldots,r$$

The matrix elements of F_r are

$$(\mathsf{F}_r)_{jk}=\langle \mathsf{F}_r e_k, e_j\rangle=r^{-1/2}e^{2\pi ijk/r},\qquad i, j=1,\ldots,r$$

Let $A=\operatorname{diag}(a_1,\ldots,a_r)$, $a_j\in\mathbb{R}$, $j=1,\ldots,r$, be a real diagonal matrix. We think of A as an observable for some r-dimensional quantum system. As usual, the eigenvalues $a_1,\ldots,a_r$ of A are the measurable values of A and the corresponding eigenvectors $e_1,\ldots,e_r$ represent the vector states in which these values are precisely attained. The *A-motion observable* is defined as $P_A=\mathsf{F}_r^* A\mathsf{F}_r$. The reason for this nomenclature is because P_A generates a one-parameter unitary group $V(t)=e^{itP_A}$ which transforms states with precise A values among themselves at certain discrete t's. More precisely, $V(2\pi j/r)e_k=e_{k+j}\ (\operatorname{mod} r)$ for $j=1,2,\ldots,r$. The matrix elements of P_A are

$$(P_A)_{jk}=r^{-1}\sum_{n=1}^{r} a_n e^{2\pi in(k-j)/r},\qquad j,k=1,\ldots,r$$

The corresponding *A-energy observable* is defined by $H_A=\alpha P_A{}^2$, where α is a constant to be determined from physical considerations. The matrix elements of H_A are

$$(H_A)_{jk}=\frac{\alpha}{r}\sum_{n=1}^{r} a_n{}^2 e^{2\pi in(k-j)/r},\qquad j,k=1,\ldots,r$$

The A-energy observable generates the *dynamical group* $U_A(t)=e^{-itH_A}$,

$t \in \mathbb{R}$. The matrix elements of $U_A(t)$ are

$$[U_A(t)]_{jk} = r^{-1} \sum_{n=1}^{r} \exp i\left[-\alpha t a_n^2 + 2\pi n(k-j)/r\right]$$

If the system is initially in the state e_k (A has value a_k), then the probability that the system will be in the state e_j (A has value a_j) at time t is

$$\begin{aligned} \mathrm{P}_{kj}(t) &= \left|\langle U_A(t)e_k, e_j\rangle\right|^2 = \left|[U_A(t)]_{jk}\right|^2 \\ &= r^{-2}\left\{r + 2\sum_{m>n=1}^{r} \cos\left[\alpha t\left(a_m^2 - a_n^2\right) + 2\pi(n-m)(k-j)/r\right]\right\} \end{aligned} \tag{1}$$

Important information about the system is also given by the average A value at time t given the initial state e_k. This is given by the formula

$$\langle A\rangle_k(t) = \langle AU_A(t)e_k, e_k\rangle = \sum_{j=1}^{r} a_j\left|[U_A(t)]_{jk}\right|^2 \tag{2}$$

3. THE QUARK SPACE

The basic quark space is $V = \mathbb{C}^{72}$. We refer the reader to Gudder (1982) for some motivation for this assumption. We may view V as the tensor product of five spaces $V = V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5$, where $V_1 = V_2 = \mathbb{C}^3$ and $V_3 = V_4 = V_5 = \mathbb{C}^2$. We call V_1 the *color space*, V_2 the *generation space*, V_3 the *charge space*, V_4 the *type space*, V_5 the *spin space*, and $V_2 \otimes V_3$ the *flavor space*. If $\{e_1, e_2, e_3\}$ and $\{f_1, f_2\}$ are the standard bases for $\mathbb{C}^3$ and $\mathbb{C}^2$, respectively, a basis for V is given by the set of *quark vectors*

$$\psi(i,j,k,m,n) = e_i \otimes e_j \otimes f_k \otimes f_m \otimes f_n, \qquad i,j = 1,2,3;\ k,m,n = 1,2$$

The usual quark classification scheme is given in Table I. Each quark also has one of the three colors r, y, b and spin up ($\uparrow$) or spin down ($\downarrow$). Our basis vectors correspond to this classification scheme as follows:

$$\psi(1,1,1,1,1) = d_r \uparrow$$

$$\psi(1,1,1,2,1) = \bar{d}_r \uparrow$$

$$\psi(2,2,2,1,2) = c_y \downarrow$$

$$\vdots$$

TABLE I

Flavor	Generation	Charge	Type
d	1	$-1/3$	Particle
u	1	$2/3$	Particle
s	2	$-1/3$	Particle
c	2	$2/3$	Particle
b	3	$-1/3$	Particle
t	3	$2/3$	Particle
$\bar{d}$	1	$1/3$	Antiparticle
$\bar{u}$	1	$-2/3$	Antiparticle
$\bar{s}$	2	$1/3$	Antiparticle
$\bar{c}$	2	$-2/3$	Antiparticle
$\bar{b}$	3	$1/3$	Antiparticle
$\bar{t}$	3	$-2/3$	Antiparticle

The following observables (operators) on V will be important in the sequel. The *color observable* C is defined by $C = \mathrm{diag}(c_1, c_2, c_3)\otimes I\otimes I\otimes I\otimes I$ where $c_1, c_2, c_3 \in \mathbb{R}$ are determined later. The *color–anticolor observable* $\bar{C} = \mathrm{diag}(c_1, c_2, c_3)\otimes I\otimes I\otimes \mathrm{diag}(1, -1)\otimes I$. The *flavor observable* $F = I\otimes \mathrm{diag}(f_1, f_2, f_3, f_4, f_5, f_6)\otimes I\otimes I$ where again $f_j \in \mathbb{R}, j = 1,\ldots,6$, are determined later. The *charge observable* $K = I\otimes I\otimes \mathrm{diag}(-1/3, 2/3)\otimes \mathrm{diag}(1, -1)\otimes I$. The *spin observable* $S = I\otimes I\otimes I\otimes I\otimes \mathrm{diag}(1/2, -1/2)$. The *generation observable* $G = I\otimes \mathrm{diag}(g_1, g_2, g_3)\otimes I\otimes \mathrm{diag}(1, -1)\otimes I$, where $g_1, g_2, g_3 \in \mathbb{R}$ are determined later. The basis vectors $\psi(i, j, k, m, n)$ are eigenvectors for the above operators with eigenvalues given by the diagonal matrix elements. It will frequently be convenient to suppress the identities in the above operators. In this case, we shall place a $\hat{}$ above the operator; for example, $\hat{C} = \mathrm{diag}(c_1, c_2, c_2)$.

4. THE COLOR OBSERVABLE

In this section we determine the eigenvalues c_1, c_2, c_3 of the color observable. Clearly, it suffices to consider the operator $\hat{C}$ on $\mathbb{C}^3$. We now make the fundamental assumptions that the c_j's are distinct and that $\hat{C}$ is traceless; that is, $c_1 + c_2 + c_3 = 0$. The reason for the first assumption is that the three colors are distinct. The reason for the second assumption is that observed particle compositions of quarks (in particular, baryons) are "colorless." Since a baryon is composed of three quarks, each of a different color, this assumption gives a total color quantum number zero.

We now consider color motion. The color dynamics $U_C(t)$ predicts that a free quark has a tendency to change color. In fact, an application of equation (1) gives

$$\mathsf{P}_{11}(t)=\tfrac{1}{3}+\tfrac{2}{9}\left[\cos\alpha t\left(c_2^2-c_1^2\right)+\cos\alpha t\left(c_3^2-c_2^2\right)+\cos\alpha t\left(c_3^2-c_1^2\right)\right]$$

It is clear that $\mathsf{P}_{11}(t)$ is periodic between 0 and 1 and that $\mathsf{P}_{11}(t_0)=1$ if and only if each of the three cosines has value 1. We must then have $\alpha t_0(c_2^2-c_1^2)=2\pi m$, $\alpha t_0(c_3^2-c_2^2)=2\pi n$, $\alpha t_0(c_3^2-c_1^2)=2\pi p$, where m, n, p are integers. We now assume that the period is a minimum. The distinctness of the c_j's then requires that $m=n=1$, $p=2$ (or a permutation of these). Solving these equations we find that within a multiplicative constant, $c_1=1$, $c_2=1+\sqrt{3}$, $c_3=-(2+\sqrt{3})$. The period then becomes $t_0=2\pi(2\sqrt{3}-3)/3\alpha$, where we determine α later (using this later value of α gives $t_0\cong 1.8\times 10^{-23}$ sec).

The graph of $\mathsf{P}_{11}(t)$ is given in Figure 1. The graphs of $\mathsf{P}_{12}(t)$ and $\mathsf{P}_{13}(t)$ are the same except for a difference in phase. We thus see that a free quark experiences a periodic color change with period t_0 given above.

If a quark is in the state e_1 (red) at time $t=0$, then applying equation (2) give the expected color value at time t:

$$\begin{aligned}\langle C\rangle_1(t)&=\sum_{j=1}^{3}c_j\left|U_C(t)_{j1}\right|^2=\sum_{j=1}^{3}c_j\mathsf{P}_{1j}(t)\\&=c_1\mathsf{P}_{11}(t)+c_2\mathsf{P}_{12}(t)-(c_1+c_2)\left[1-\mathsf{P}_{11}(t)-\mathsf{P}_{12}(t)\right]\\&=(2c_1+c_2)\mathsf{P}_{11}(t)+(c_1+2c_2)\mathsf{P}_{12}(t)-(c_1+c_2)\end{aligned}$$

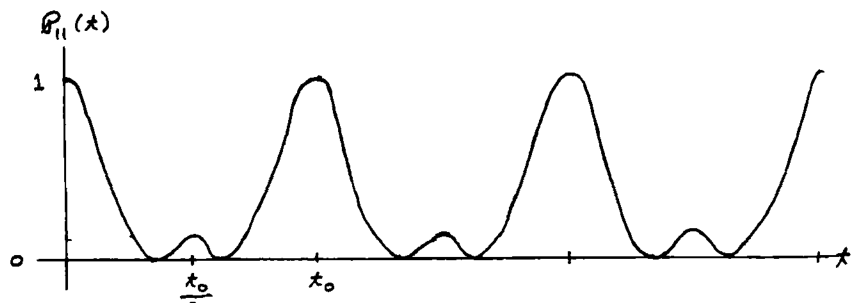

Fig. 1. Graph of $\mathsf{P}_{11}(t)$.

It is easy to check that

$$\frac{1}{t_0}\int_0^{t_0}\mathsf{P}_{11}(t)\,dt=\frac{1}{t_0}\int_0^{t_0}\mathsf{P}_{12}(t)\,dt=\frac{1}{3}$$

Hence, the average color value over a single period becomes

$$\frac{1}{t_0}\int_0^{t_0}\langle C\rangle_1(t)\,dt=\frac{1}{3}(2c_1+c_2)+\frac{1}{3}(c_1+2c_2)-(c_1+c_2)=0$$

We close this section by computing the color motion observable $P_{\hat{C}}$ and the color energy observable $\alpha P_{\hat{C}}^2$. According to the formula given in Section 2 we have

$$(P_{\hat{C}})_{jk}=(1/3)\sum_{n=1}^{3}c_n e^{2\pi in(k-j)/3},\qquad j,k=1,2,3$$

It follows that

$$P_{\hat{C}}=\begin{bmatrix} 0 & -1-\frac{\sqrt{3}}{2}-\frac{1}{2}i & -1-\frac{\sqrt{3}}{2}+\frac{1}{2}i \\ -1-\frac{\sqrt{3}}{2}+\frac{1}{2}i & 0 & -1-\frac{\sqrt{3}}{2}-\frac{1}{2}i \\ -1-\frac{\sqrt{3}}{2}-\frac{1}{2}i & -1-\frac{\sqrt{3}}{2}+\frac{1}{2}i & 0 \end{bmatrix}$$

Similarly,

$$(H_{\hat{C}})_{jk}=\frac{\alpha}{3}\sum_{n=1}^{3}c_n{}^2 e^{2\pi in(k-j)/3},\qquad j,k=1,2,3$$

Hence,

$$H_{\hat{C}}=\alpha\begin{bmatrix} 4+2\sqrt{3} & \frac{3}{2}+\sqrt{3}-\left(\frac{\sqrt{3}}{2}+1\right)i & \frac{3}{2}+\sqrt{3}+\left(\frac{\sqrt{3}}{2}+1\right)i \\ \frac{3}{2}+\sqrt{3}+\left(\frac{\sqrt{3}}{2}+1\right)i & 4+2\sqrt{3} & \frac{3}{2}+\sqrt{3}-\left(\frac{\sqrt{3}}{2}+1\right)i \\ \frac{3}{2}+\sqrt{3}-\left(\frac{\sqrt{3}}{2}+1\right)i & \frac{3}{2}+\sqrt{3}+\left(\frac{\sqrt{3}}{2}+1\right)i & 4+2\sqrt{3} \end{bmatrix}$$

We then assume that $P_C=P_{\hat{C}}\otimes I\otimes I\otimes I\otimes I$ and $H_C=H_{\hat{C}}\otimes I\otimes I\otimes I\otimes I$.

5. THE QUARK TENSOR SPACE

The *quark tensor space* is defined as

$$TV = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$
$$= \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$

The normalized vectors in TV represent the vector states of all possible combinations of quarks. For $\phi \in TV$ we write $\phi = \phi_0 \oplus \phi_1 \oplus \cdots$, where $\phi_n \in V^{\otimes n}$. Define the projection P_n: $TV \to TV$ by

$$P_n \phi = 0 \oplus 0 \oplus \cdots \oplus \phi_n \oplus 0 \oplus \cdots, \qquad n = 0, 1, \ldots$$

If A: $V \to V$ is an observable we define $\Gamma(A)$: $TV \to TV$ by

$$\Gamma(A) = I \oplus A \oplus (A \otimes I + I \otimes A) \oplus (A \otimes I \otimes I + I \otimes A \otimes I + I \otimes I \otimes A) \oplus \cdots$$
$$= \Gamma^0(A) \oplus \Gamma^1(A) \oplus \Gamma^2(A) \oplus \cdots$$

We call $\phi \in TV$ a *color vector* if $\Gamma(\bar{C}) P_n \phi = \Gamma^n(\bar{C}) \phi_n = 0$ for $n = 2, 3, 4, \ldots$. Except for the quarks themselves, normalized color vectors represent vector states of "colorless" particles. The set of color vectors forms a closed subspace of TV and is denoted by

$$(TV)_C = \mathbb{C} \oplus V \oplus V_C^2 \oplus V_C^3 \oplus \cdots$$

where

$$V_C^n = \{\phi_n \in V^{\otimes n} : \Gamma^n(\bar{C}) \phi_n = 0\}$$

Let $\phi_n = \psi_1 \otimes \cdots \otimes \psi_n \in V_C^n$, $n \geqslant 2$, where the ψ_j's are quark vectors. If $\bar{C}\psi_j = -\bar{C}\psi_k$ we call $\{\psi_j, \psi_k\}$ a *meson pair* in ϕ_n. Of course, in a meson pair one of the vectors represents a quark and the other an antiquark. If ψ_i, ψ_j, and ψ_k represent quarks and $\bar{C}(\psi_i + \psi_j + \psi_k) = 0$, then we call $\{\psi_i, \psi_j, \psi_k\}$ a *baryon triple* in ϕ_n. If ψ_i, ψ_j, and ψ_k represent antiquarks and $\bar{C}(\psi_i + \psi_j + \psi_k) = 0$, then we call ψ_i, ψ_j, ψ_k an *antibaryon triple* in ϕ_n.

> *Theorem.* Let $\phi \in TV$. Then $\phi \in (TV)_C$ if and only if ϕ_n is a linear combination of terms each of which is composed of m meson pairs, p baryon triples, and q antibaryon triples ($m, p, q \geqslant 0$ depend on the term, $2m + 3p + 3q = n$), $n = 2, 3, \ldots$.

Proof. Sufficiency is clear. For necessity, let $\phi \in (TV)_C$ and suppose ϕ_n is a product of quark vectors $\phi_n = \psi_1 \otimes \cdots \otimes \psi_n \in V_C^n$. Let $\bar{C}\psi_j = a_j \psi_j$,

$j=1,\ldots,n$. Since $\Gamma^n(\bar{C})\phi_n=0$, we have $\sum_{j=1}^n a_j=0$. We can assume without loss of generality that $\psi_1,\ldots,\psi_k$ represent quarks and $\psi_{k+1},\ldots,\psi_n$ represent antiquarks. We can also assume without loss of generality that $a_{k+1}=-a_j$ for $j=1,\ldots,m$ and some $m\geqslant 0$. Then $\{\psi_j,\psi_{k+j}\}$, $j=1,\ldots,m$, form m meson pairs. We now consider the remaining quark vectors $\psi_{m+1},\ldots,\psi_k$; $\psi_{k+m+1},\ldots,\psi_n$. We then have

$$\sum_{j=m+1}^{k} a_j+\sum_{j=k+m+1}^{n} a_j=0$$

Let p_i be the number of c_i's, $i=1,2,3$ (the c_i's are the color eigenvalues) and q_i the number of $-c_i$'s, $i=1,2,3$, among these a_j's. Notice that $q_i=0$ if $p_i\neq 0$, $i=1,2,3$.

Case 1. $p_1,p_2,p_3\neq 0$. Then $q_1=q_2=q_3=0$ and $p_1+p_2(1+\sqrt{3})-p_3(2+\sqrt{3})=0$. It follows that $p_1+p_2-2p_3=0$ and $p_2-p_3=0$. Hence, $p_1=p_2=p_3$. We then have m meson pairs, p_1 baryon triples, and 0 antibaryon triples.

Case 2. $p_1=p_2=p_3=0$. As in Case 1, $q_1=q_2=q_3$. We then have m meson pairs, q_1 antibaryon triples, and 0 baryon triples.

Case 3. $p_1=0$, $p_2,p_3\neq 0$. Then $q_2=q_3=0$ and $p_2(1+\sqrt{3})-p_3(2+\sqrt{3})=0$. It follows that $q_1=0$ and $p_2-2p_3=0$, $p_2-p_3=0$. Hence, $p_2=p_3=0$, a contradiction.

Case 4. $p_2=0$, $p_1,p_3\neq 0$. As in Case 3, this is impossible.

Case 5. $p_3=0$, $p_1,p_2\neq 0$. As in Case 3, this is impossible.

In Case 1, the composition consisting of m meson pairs and p_1 baryon triples is only one of possibly many compositions. In general, the m meson pairs can be redistributed to form m_0 meson pairs ($m_0<m$), p_0 baryon triples, and p_0 antibaryon triples, where $m_0+3p_0=m$. This would result in a total of m_0 meson pairs, p_0+p_1 baryon triples and p_0 antibaryon triples. Similarly, for Case 2, in general we would have m_0 meson pairs, p_0 baryon triples, and p_0+q_1 antibaryon triples.

More generally, suppose $\phi_n\in V_C{}^n$ and $\phi_n=\sum_{j=1}^n b_j\chi_j$, where $b_j\neq 0$, χ_j's are product quark vectors. Then each χ_j is an eigenvector of $\Gamma^n(\bar{C})$ with eigenvalue λ_j, say. Hence,

$$0=\Gamma^n(\bar{C})\phi_n=\sum_{j=1}^{r} b_j\lambda_j\chi_j$$

Since the χ_j's are linearly independent, we have $\lambda_j=0$, $j=1,\ldots,r$. Then as before χ_j consists of m meson pairs, p baryon triples, and q antibaryon triples, $m,p,q\geqslant 0$, $2m+3p+3q=n$. ■

Corollary. If $\phi_2 \in V_C^3$, then ϕ_2 is a linear combination of meson pairs. If $\phi_3 \in V_C^3$, then ϕ_3 is a linear combination of baryon triples and antibaryon triples.

It appears that the following *color condition* is a law of nature. Of the normalized vectors in TV only the color vectors are observable physical vector states. It follows from the color condition and the above theorem that quarks can only combine as mesons, baryons, antibaryons, and collections of these three.

6. BARYON SPACE

We call V_C^3 the *baryon space*. Certain unit vectors of V_C^3 represent ground states (zero orbital angular momentum) of baryons and antibaryons. A basis for V_C^3 is the set of quark product vectors

$$\psi(i, j, k, m, n) \otimes \psi(i', j', k', m', n') \otimes \psi(i'', j'', k'', m'', n'')$$

where i, i', and i'' are distinct and $m = m' = m''$. The dimension of V_C^3 is

$$\begin{aligned} \dim V_C^3 &= (3\times 3\times 2\times 2\times 2)\times(2\times 3\times 2\times 1\times 2)\times(1\times 3\times 2\times 1\times 2) \\ &= 20{,}736. \end{aligned}$$

Half of this dimension represents baryons and the other half represents antibaryons. Since there is a one-to-one correspondence between baryons and their corresponding antibaryons, we shall only consider the 10,368 dimensional space of baryons ($m = 1$). Since a baryon is "colorless," there is no physical way of distinguishing between the six possible color permutations. For example,

$$\psi(1, j, k, 1, n) \otimes \psi(2, j', k', 1, n') \otimes \psi(3, j'', k'', 1, n'') \tag{3}$$

and

$$\psi(2, j, k, 1, n) \otimes \psi(3, j', k', 1, n') \otimes \psi(1, j'', k'', 1, n'')$$

are physically equivalent. Hence, we need only consider the 1728-dimensional subspace spanned by the vectors in equation (3). If we disregard spin which can be treated in the usual way, we obtain $1728/8 = 216$ baryon vectors.

In order to obtain actual physical states, these baryon vectors must be properly symmetrized (S), antisymmetrized (A), or mixed symmetrized (M). It turns out that there are 56 S states, 20 A states, and 140 M states (Close, 1979). A natural classification of baryons can be given in terms of symmetrization and generation. We define the generation observable G to be

$$G = I \otimes \operatorname{diag}(-5/3, -2/3, 7/3) \otimes I \otimes \operatorname{diag}(1, -1) \otimes I$$

We chose these numbers since they make $\hat{G}$ traceless and give the smallest integer generation numbers for baryons and mesons. The *baryon generation observable* is $\Gamma^3(G)$ and the *generation number* of a baryon is the eigenvalue of that baryon under $\Gamma^3(G)$. Tables II, III, and IV display this classification where each baryon is represented by its quark composition. In Table IV there are four mixed states for each baryon at the 0 generation level and two or four mixed states for each baryon at the other levels (Close, 1979). The well-known mixed octet and symmetric decuplet are underlined in Tables IV and II, respectively. The symmetric states have spin $3/2$ and the antisymmetric and mixed states have spin $1/2$.

TABLE II. 56 Symmetric States.

Generation number	Quark composition							
7			*bbb*	*bbt*	*btt*	*ttt*		
4		*sbb*	*stt*	*cbb*	*ctt*	*sbt*	*cbt*	
3		*dbb*	*dtt*	*dbt*	*ubb*	*utt*	*ubt*	
1		*ssb*	*sst*	*ccb*	*cct*	*scb*	*sct*	
0	*dcb*	*dct*	*dsb*	*dst*	*ucb*	*uct*	*usb*	*ust*
−1		*ddb*	*ddt*	*uub*	*uut*	*dub*	*dut*	
−2			*ssc*	$\underline{sss}$	*scc*	*ccc*		
−3		*dcc*	$\underline{dss}$	$\underline{uss}$	*ucc*	*dsc*	*usc*	
−4		*ddc*	$\underline{dds}$	$\underline{uus}$	$\underline{dus}$	*uuc*	*duc*	
−5			$\underline{ddd}$	$\underline{ddu}$	$\underline{duu}$	$\underline{uuu}$		

TABLE III. 20 Antisymmetric States

Generation number	Quark composition							
4				*sbt*	*cbt*			
3				*dbt*	*ubt*			
1				*scb*	*sct*			
0	*dcb*	*dct*	*dsb*	*dst*	*ucb*	*uct*	*usb*	*ust*
−1				*dub*	*dut*			
−3				*dcs*	*ucs*			
−4				*dus*	*duc*			

TABLE IV. 140 Mixed States

Generation number	Quark composition							
7				$bbt(2)$	$btt(2)$			
4		$sbt(4)$	$sbb(2)$	$ctt(2)$	$cbb(2)$	$stt(2)$	$cbt(4)$	
3		$dbt(4)$	$dbb(2)$	$dtt(2)$	$ubb(2)$	$utt(2)$	$ubt(4)$	
1		$sct(4)$	$ssb(2)$	$sst(2)$	$ccb(2)$	$cct(2)$	$scb(4)$	
0	$dcb(4)$	$dct(4)$	$dsb(4)$	$dst(4)$	$ucb(4)$	$uct(4)$	$usb(4)$	$ust(4)$
-1		$dut(4)$	$ddb(2)$	$ddt(2)$	$uub(2)$	$uut(2)$	$dub(4)$	
-2				$ssc(2)$	$scc(2)$			
-3		$usc(4)$	$ucc(2)$	$\underline{dss}\,(2)$	$\underline{uss}\,(2)$	$dcc(2)$	$dsc(4)$	
-4		$duc(4)$	$ddc(2)$	$\underline{dds}\,(2)$	$\underline{uus}\,(2)$	$uuc(2)$	$\underline{dus}\,(4)$	
-5				$\underline{ddu}\,(2)$	$\underline{duu}\,(2)$			

7. MESON SPACE

We call V_C^2 the *meson space*. Certain unit vectors of V_C^2 represent ground states of mesons. A basis for V_C^2 is the set of quark product vectors

$$\psi(i, j, k, m, n) \otimes \psi(i', j', k', m', n')$$

where $i = i'$, $m \neq m'$. The dimension of V_C^2 is

$$\dim V_C^2 = (3 \times 3 \times 2 \times 2 \times 2) \times (1 \times 3 \times 2 \times 1 \times 2) = 864.$$

Since $i = 1, 2, 3$ are physically equivalent, we need only consider the 288-dimensional subspaces spanned by the vectors

$$\psi(1, j, k, m, n) \times \psi(1, j', k', m', n'), \qquad m \neq m'$$

TABLE V. 36 Symmetric States

Generation number	Quark composition											
4					$b\bar{d}$	$b\bar{u}$	$t\bar{d}$	$t\bar{u}$				
3					$b\bar{s}$	$b\bar{c}$	$t\bar{s}$	$t\bar{c}$				
1					$c\bar{d}$	$\underline{s\bar{d}}$	$\underline{s\bar{u}}$	$c\bar{u}$				
0	$t\bar{t}$	$t\bar{b}$	$b\bar{t}$	$b\bar{b}$	$\underline{d\bar{u}}$	$\underline{u\bar{d}}$	$\underline{u\bar{u}}$	$\underline{s\bar{s}}$	$\underline{d\bar{d}}$	$s\bar{c}$	$c\bar{s}$	$c\bar{c}$
-1					$d\bar{c}$	$\underline{d\bar{s}}$	$\underline{u\bar{s}}$	$u\bar{c}$				
-3					$s\bar{b}$	$s\bar{t}$	$c\bar{b}$	$c\bar{t}$				
-4					$d\bar{b}$	$d\bar{t}$	$u\bar{b}$	$u\bar{t}$				

If we disregard spin we have $288/4 = 72$ meson vectors. These may be symmetrized or antisymmetrized giving 36 symmetric states and 36 antisymmetric states. The generation number of a meson is the eigenvalue of that meson under $\Gamma^2(G)$. Table V displays the symmetric states classified according to generation number, where each meson is represented by its quark decomposition. The well-known meson nonet is underlined. The antisymmetric classification is similar. The symmetric states have spin 0 and the antisymmetric states have spin 1.

8. THE FLAVOR OBSERVABLE

We now determine the eigenvalues, $f_1, \ldots, f_6$ of the flavor observable F. We do this using the approximately known masses of various quarks. Some of these masses can be estimated using tables such as Table VI. Using Table VI and estimates for b and t mesons we obtain the rough approximations given in Table VII.

We assume that the free quark Hamiltonian has the following form:

$$H^{(1)} = \alpha P_C{}^2 + \gamma F^2 \tag{4}$$

Instead of viewing $H^{(1)}$ as the energy observable, it is more convenient to regard it as the equivalent mass observable. Equation (4) states that the mass of a quark is the sum of its color energy mass and its "rest" mass. The color energy mass results from the fact that the quark's color is continually

TABLE VI. Particle Masses

Particle	Quark composition	Mass, MeV
p	uud	938
Σ^+	uus	1189
ψ	$c\bar{c}$	3100

TABLE VII. Quark Masses

Quark	Mass, MeV
d	320
u	320
s	500
c	1500
b	4000
t	16000

changing, giving rise to a kind of "kinetic" energy, and the "rest" mass is the mass a quark would have if its color were not changing. The basic assumption in equation (4) is that the rest mass is proportional to the square of the flavor. If $\psi \in V$ is a quark vector state, the expected mass in state ψ becomes

$$\langle H^{(1)}\psi, \psi\rangle = \alpha\langle P_C{}^2\psi, \psi\rangle + \gamma\langle F^2\psi, \psi\rangle$$

Notice that the diagonal elements of $P_C{}^2$ are equal and let

$$a = \alpha\langle \hat{P}_C{}^2 r, r\rangle = \alpha\langle \hat{P}_C{}^2 y, y\rangle = \alpha\langle \hat{P}_C{}^2 b, b\rangle$$

We then have the following approximate formulas:

$$320 = \langle H^{(1)}d, d\rangle = a + \gamma f_1{}^2$$

$$320 = \langle H^{(1)}u, u\rangle = a + \gamma f_2{}^2$$

$$500 = \langle H^{(1)}s, s\rangle = a + \gamma f_3{}^2$$

$$1500 = \langle H^{(1)}c, c\rangle = a + \gamma f_4{}^2$$

$$4000 = \langle H^{(1)}b, b\rangle = a + \gamma f_5{}^2$$

$$16000 = \langle H^{(1)}t, t\rangle = a + \gamma f_6{}^2$$

Now the mass of the π^0 meson, $\pi^0 = u\bar{u}$, is approximately 135 MeV. Since the masses of u and $\bar{u}$ are about 320 MeV, this indicates that about 255 MeV from the quark and also the antiquark is converted into quark–antiquark interaction energy which we assume consists of color energy. The remaining 65 MeV of the quark and antiquark contribute to the meson mass. For this reason, we assume that $a \cong 255$, $\gamma \cong 65$, and $f_1 \cong f_2 \cong \pm 1$. Solving the remaining four equations gives

$$f_3 \cong [(500 - 255)/65]^{1/2} = \pm 1.94$$

$$f_4 \cong [(1500 - 255)/65]^{1/2} = \pm 4.38$$

$$f_5 \cong [(4000 - 255)/65]^{1/2} = \pm 7.59$$

$$f_6 \cong [(1600 - 255)/65]^{1/2} = \pm 15.56$$

The above equations do not determine the signs of the f_i's, $i=1,\ldots,6$. Also, these numbers are only rough approximations. Our later mass formulas give the best agreement if we choose the signs $+,+,-,-,+,+$. Moreover, assuming that nature gives a regular pattern we postulate that $f_1 = 1\frac{1}{32}$, $f_2 = 1, f_3 = -2$, $f_4 = -4$, $f_5 = 8$, $f_6 = 16$. We have assumed that $f_1 = 1\frac{1}{32}$ to take account of small mass differences between particles containing d quarks and those containing u quarks.

We now compute the flavor motion observable P_F. We assume that P_F commutes with the generation observable G; that is, generation is a constant of the flavor motion. This is indicated by the fact that quarks seem to prefer to stay in their own generation. In flavor space, G has the form $G = \operatorname{diag}(-5/3, -5/3, -2/3, -2/3, 7/3, 7/3)$. For P_F to commute with G we conclude that $\hat{P}_F$ has the form

$$\hat{P}_F = \mathsf{F}_2{}^*\operatorname{diag}(1\tfrac{1}{32}, 1)\mathsf{F}_2 \oplus \mathsf{F}_2{}^*\operatorname{diag}(-2, -4)\mathsf{F}_2 \oplus \mathsf{F}_2{}^*\operatorname{diag}(8, 16)\mathsf{F}_2$$

A simple computation gives

$$\hat{P}_F = \begin{bmatrix} 1\frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 \\ -\frac{1}{64} & 1\frac{1}{64} & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 4 \\ 0 & 0 & 0 & 0 & 4 & 12 \end{bmatrix}$$

9. BARYON HAMILTONIAN

In Section 8 we proposed a free quark Hamiltonian $H^{(1)}$ given by equation (4). In this section we shall propose a baryon Hamiltonian $H^{(3)}$. By computing the mass expectations of known baryons in terms of this Hamiltonian, we shall determine the two constants in equation (4) as well as four others appearing in $H^{(3)}$. We then compare our predicted results with other baryon masses. Since $H^{(3)}$ is the Hamiltonian for three interacting quarks, we should expect some interaction terms which do not appear in $H^{(1)}$.

Our proposed baryon Hamiltonian is

$$\begin{aligned} H^{(3)} = {} & \alpha\Gamma^3(P_C{}^2) + \beta(P_C \otimes P_C \otimes I + P_C \otimes I \otimes P_C + I \otimes P_C \otimes P_C) + \gamma\Gamma^3(F^2) \\ & - \delta(F \otimes F \otimes I + F \otimes I \otimes F + I \otimes F \otimes F) - \mu F \otimes F \otimes F - \nu K \otimes K \otimes K \end{aligned}$$

The first term is a color energy term, the second is a color motion interaction between pairs of quarks, the third is a rest mass term, the fourth is a flavor interaction between pairs of quarks, and the fifth and sixth are a flavor interaction and a charge interaction, respectively, among all three quarks. The mass expectation for a baryon in the vector state ψ is given by $\langle H^{(3)}\psi, \psi\rangle$.

By computing the mass expectations for the six baryons $n, p, \Sigma^+, \Xi^0, \Delta^{++}, \Omega^-$ we obtain the following values for the constants:

$$\begin{aligned} \alpha &= 35.50, & \delta &= 11.56 \\ \beta &= 26.46, & \mu &= 2.33 \\ \gamma &= 58.77, & \nu &= 7.19 \end{aligned}$$

Using these values for α and γ we obtain the following quark masses:

$$\begin{aligned} \langle H^{(1)}d, d\rangle &= \alpha\langle P_C^2 d, d\rangle + \gamma\langle F^2 d, d\rangle \\ &= 265.05 + 62.50 = 327.55 \\ \langle H^{(1)}u, u\rangle &= 265.05 + 58.77 = 323.82 \\ \langle H^{(1)}s, s\rangle &= 265.05 + 235.08 = 500.13 \\ \langle H^{(1)}c, c\rangle &= 265.05 + 940.32 = 1205.37 \\ \langle H^{(1)}b, b\rangle &= 265.05 + 3761.28 = 4026.33 \\ \langle H^{(1)}t, t\rangle &= 265.05 + 15045.12 = 15310.17 \end{aligned}$$

Instead of solving the six baryon equations we shall work backwards and show that the six constants give the correct mass values.

We first consider the neutron, which is given by the mixed state

$$n = (2)^{-1/2}(u_r\uparrow \otimes d_y\downarrow \otimes d_b\downarrow - d_y\downarrow \otimes u_r\uparrow \otimes d_b\downarrow)$$

The neutron mass expectation becomes

$$\begin{aligned} \langle H^{(3)}n, n\rangle = 3\alpha\langle P_C^2 u_r\uparrow, u_r\uparrow\rangle \\ &+ \gamma\left[\langle F^2 u_r\uparrow, u_r\uparrow\rangle + \langle F^2 d_y\downarrow, d_y\downarrow\rangle + \langle F^2 d_b\downarrow, d_b\downarrow\rangle\right] \\ &- \delta\left[\langle Fu_r\uparrow, u_r\uparrow\rangle\langle Fd_y\downarrow, d_y\downarrow\rangle + \langle Fu_r\uparrow, u_r\uparrow\rangle\right. \\ &\quad \times\langle Fd_b\downarrow, d_b\downarrow\rangle + \langle Fd_y\downarrow, d_y\downarrow\rangle\langle Fd_b\downarrow, d_b\downarrow\rangle \end{aligned}$$

$$
\begin{aligned}
&-\mu\langle Fu_r\uparrow, u_r\uparrow\rangle\langle Fd_y\downarrow, d_y\downarrow\rangle\langle Fd_b\downarrow, d_b\downarrow\rangle\\
&-\nu\langle Ku_r\uparrow, u_r\uparrow\rangle\langle Kd_y\downarrow, d_y\downarrow\rangle\langle Kd_b\downarrow, d_b\downarrow\rangle\\
&=3\alpha\langle \hat{P}_C{}^2 r, r\rangle+\gamma\left[\langle \hat{F}^2u,u\rangle+2\langle \hat{F}^2d,d\rangle\right]\\
&-\delta\left[2\langle \hat{F}u,u\rangle\langle \hat{F}d,d\rangle+\langle \hat{F}d,d\rangle^2\right]\\
&-\mu\langle \hat{F}u,u\rangle\langle \hat{F}d,d\rangle^2-\nu\langle \hat{K}u,u\rangle\langle \hat{K}d,d\rangle^2\\
&=3(4+2\sqrt{3})\alpha+\left[1+2\left(1\tfrac{1}{32}\right)^2\right]\gamma-\left[2\left(\tfrac{1}{32}\right)+\left(1\tfrac{1}{32}\right)^2\right]\delta\\
&-\left(1\tfrac{1}{32}\right)^2\mu-\left(\tfrac{2}{3}\right)\left(-\tfrac{1}{3}\right)^2\nu=939.55
\end{aligned}
$$

Similarly, the proton

$$p=(2)^{-1/2}(d_r\uparrow\otimes u_y\downarrow\otimes u_b\downarrow-u_r\downarrow\otimes d_r\uparrow\otimes u_b\downarrow)$$

has mass expectation

$$
\begin{aligned}
\langle H^{(3)}p,p\rangle&=3(4+2\sqrt{3})\alpha+\left[2+\left(1\tfrac{1}{32}\right)^2\right]\gamma-\left[2\left(\tfrac{1}{32}\right)+1\right]\delta-\left(1\tfrac{1}{32}\right)\mu\\
&\quad-\left(\tfrac{2}{3}\right)^2\left(-\tfrac{1}{3}\right)\nu\\
&=938.23
\end{aligned}
$$

The Σ^+ baryon

$$\Sigma^+=(2)^{-1/2}(s_r\uparrow\otimes u_y\downarrow\otimes u_b\downarrow-u_y\downarrow\otimes s_r\uparrow\otimes u_b\downarrow)$$

has mass expectation

$$
\begin{aligned}
\langle H^{(3)}\Sigma^+,\Sigma^+\rangle&=3(4+2\sqrt{3})\alpha+6\gamma+3\delta+2\mu-\left(-\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^2\nu\\
&=1188
\end{aligned}
$$

The Ξ^0 baryon

$$\Xi^0=(2)^{-1/2}(u_r\uparrow\otimes s_y\downarrow\otimes s_b\downarrow-s_y\downarrow\otimes u_r\uparrow\otimes s_b\downarrow)$$

has mass expectation

$$
\begin{aligned}
\langle H^{(3)}\Xi^0,\Xi^0\rangle&=3(4+2\sqrt{3})\alpha+9\gamma-4\mu-\tfrac{2}{3}\left(-\tfrac{1}{3}\right)^2\nu\\
&=1314
\end{aligned}
$$

The Δ^{++} baryon is given by the symmetric state

$$\Delta^{++}=(6)^{-1/2}\big(u_r\uparrow\otimes u_y\uparrow\otimes u_b\uparrow+u_r\uparrow\otimes u_b\uparrow\otimes u_y\uparrow+u_y\uparrow\otimes u_r\uparrow\otimes u_b\uparrow$$
$$+u_y\uparrow\otimes u_b\uparrow\otimes u_r\uparrow+u_b\uparrow\otimes u_r\uparrow\otimes u_y\uparrow+u_b\uparrow\otimes u_y\uparrow\otimes u_r\uparrow\big)$$

The Δ^{++} mass expectation becomes

$$\begin{aligned}\langle H^{(3)}\Delta^{++},\Delta^{++}\rangle&=3\alpha\langle\hat{P}_C{}^2r,r\rangle+3\beta\left|\langle\hat{P}_Cr,y\rangle\right|^2+3\gamma-3\delta-\mu-\left(\tfrac{2}{3}\right)^3\nu\\&=3(4+2\sqrt{3})\alpha+3\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta+3\gamma-3\delta-\mu-\left(\tfrac{2}{3}\right)^2\nu\\&=1228\end{aligned}$$

Similarly, the $\Omega^-=sss$ baryon has mass expectation

$$\begin{aligned}\langle H^{(3)}\Omega^-,\Omega^-\rangle&=3(4+2\sqrt{3})\alpha+3\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\\&\quad+12\gamma-12\delta+8\mu-\left(-\tfrac{1}{3}\right)^2\nu\\&=1677\end{aligned}$$

All of these values are within 0.3% of known experimental values. (The accuracy can be improved by determining the constant values to more significant figures.) The accuracy is maintained for the baryons $\Sigma^0=uds$, $\Sigma^-=dds$, $\Xi^-=dss$, $\Delta^+=uud$, $\Delta^0=udd$, $\Delta^-=ddd$:

$$\begin{aligned}\langle H^{(3)}\Sigma^0,\Sigma^0\rangle&=3(4+2\sqrt{3})\alpha+\left[5+\left(1\tfrac{1}{32}\right)^2\right]\gamma+\left(2+1\tfrac{1}{32}\right)\delta+2\left(1\tfrac{1}{32}\right)\mu\\&\quad-\tfrac{2}{3}\left(-\tfrac{1}{3}\right)^2\nu\\&=1191\end{aligned}$$

$$\begin{aligned}\langle H^{(3)}\Sigma^-,\Sigma^-\rangle&=3(4+2\sqrt{3})\alpha+\left[4+2\left(1\tfrac{1}{32}\right)^2\right]\gamma+\left[4\left(1\tfrac{1}{32}\right)-\left(1\tfrac{1}{32}\right)^2\right]\delta\\&\quad+2\left(1\tfrac{1}{32}\right)^2\mu-\left(-\tfrac{1}{3}\right)^2\nu\\&=1196\end{aligned}$$

$$\begin{aligned}\langle H^{(3)}\Xi^-,\Xi^-\rangle&=3(4+2\sqrt{3})\alpha+\left[8+\left(1\tfrac{1}{32}\right)^2\right]\gamma+\left[4\left(\tfrac{1}{32}\right)-4\right]\delta-4\left(1\tfrac{1}{32}\right)\mu\\&\quad-\left(-\tfrac{1}{3}\right)^3\nu\\&=1320\end{aligned}$$

$$\langle H^{(3)}\Delta^+, \Delta^+\rangle = 3(4+2\sqrt{3})\alpha + 3\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + \left[2+(1\tfrac{1}{32})^2\right]\gamma$$

$$-\left[2(\tfrac{1}{32})+1\right]\delta - (1\tfrac{1}{32})\mu - (\tfrac{2}{3})^2(-\tfrac{1}{3})\nu$$

$$= 1234$$

$$\langle H^{(3)}\Delta^0, \Delta^0\rangle = 3(4+2\sqrt{3})\alpha + 3\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + \left[1+2(1\tfrac{1}{32})^2\right]\gamma$$

$$-\left[2(\tfrac{1}{32})+(1\tfrac{1}{32})^2\right]\delta - (1\tfrac{1}{32})^2\mu - (\tfrac{2}{3})(-\tfrac{1}{3})^2\nu$$

$$= 1236$$

$$\langle H^{(3)}\Delta^-, \Delta^-\rangle = 3(4+2\sqrt{3})\alpha + 3\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + 3(1\tfrac{1}{32})^2\gamma - 3(1\tfrac{1}{32})^2\delta$$

$$-(1\tfrac{1}{32})^3\mu - (-\tfrac{1}{3})^3\nu$$

$$= 1240$$

The color symmetrization for the symmetric states $\Delta^{++} = uuu$ and $\Omega^- = sss$ were straightforward. However, the proper color symmetrization for other symmetric states is not so clear. We shall symmetrize relative to color as follows. If all three quarks are in the same generation, completely symmetrize relative to color. If they are not all in the same generation, symmetrize only the ones in the same generation as in the following example of $\Sigma^{*+} = uus$ (we have deleted the normalization):

$$\Sigma^{*+} = u_r u_y s_b + u_y u_r s_b + u_r u_b s_y + u_b u_r s_y + u_b s_r u_y + u_y s_r u_b + u_r s_b u_y$$

$$+ u_y s_b u_r + s_y u_r u_b + s_y u_b u_r + s_r u_b u_y + s_r u_y u_b$$

With this color symmetrization, our mass expectations are again within 0.3% of the experimental values:

$$\langle H^{(3)}\Sigma^{*+}, \Sigma^{*+}\rangle = 3(4+2\sqrt{3})\alpha + 2\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + 6\gamma + 3\delta$$

$$+2\mu - (-\tfrac{1}{3})(\tfrac{2}{3})^2\nu$$

$$= 1385$$

In a similar way, for $\Sigma^{*0} = uds$, $\Sigma^{*-} = dds$, $\Xi^{*0} = uss$, $\Xi^{*-} = dss$, we obtain

$$\langle H^{(3)}\Sigma^{*0}, \Sigma^{*0}\rangle = 1388$$

$$\langle H^{(3)}\Sigma^{*-}, \Sigma^{*-}\rangle = 1393$$

$$\langle H^{(3)}\Xi^{*0}, \Xi^{*0}\rangle = 1512$$

$$\langle H^{(3)}\Xi^{*-}, \Xi^{*-}\rangle = 1518$$

For Λ, the antisymmetric uds, we first antisymmetrize relative to flavor to obtain

$$uds + sud + dsu - [usd + sdu + dus]$$

We now completely symmetrize the first three terms relative to color and symmetrize the bracketed term as follows:

$$u_r s_y d_b + u_b s_r d_y + u_y d_b s_r + s_r d_b u_y + s_b d_y u_r + s_y d_r u_b$$
$$+ d_r u_y s_b + d_b u_r s_y + d_y u_r s_b$$

We then have

$$\begin{aligned}\langle H^{(3)}\Lambda, \Lambda\rangle &= 3(4+2\sqrt{3})\alpha + (2\tfrac{4}{27})\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + \left[5+(1\tfrac{1}{32})^3\right]\gamma \\ &\quad + \left[2+(1\tfrac{1}{32})\right]\delta + 2(1\tfrac{1}{32})\mu - (\tfrac{2}{3})(-\tfrac{1}{3})^2\nu \\ &= 1403\end{aligned}$$

For Λ^0 we take the mixed state

$$\begin{aligned}(5)^{-1/2}\big(&u_r\uparrow\otimes d_y\downarrow\otimes s_b\downarrow - u_r\uparrow\otimes d_b\downarrow\otimes s_y\downarrow - d_y\downarrow\otimes u_r\uparrow\otimes s_b\downarrow \\ &+ d_y\downarrow\otimes u_b\uparrow\otimes s_r\downarrow - s_b\downarrow\otimes d_y\downarrow\otimes u_r\uparrow\big)\end{aligned}$$

We then have

$$\begin{aligned}\langle H^{(3)}\Lambda^0, \Lambda^0\rangle &= 3(4+2\sqrt{3})\alpha - \tfrac{4}{5}\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + \left[5+(1\tfrac{1}{32})^2\right]\gamma \\ &\quad + \left[2+(1\tfrac{1}{32})\right]\delta + 2(1\tfrac{1}{32})\mu - \tfrac{2}{3}(-\tfrac{1}{3})^2\nu \\ &= 1112\end{aligned}$$

10. MESON HAMILTONIAN

As in the baryon Hamiltonian, the meson Hamiltonian $H^{(2)}$ has six terms. However, the counterpart of the $\mu F\otimes F\otimes F$ term is unnecessary and as discussed in Section 8, the $\Gamma^2(P_C{}^2)$ term vanishes due to a quark–antiquark color energy cancellation. These terms are replaced by two quark–antiquark interactions. We propose the following meson Hamiltonian

$$H^{(2)}=6\beta(P_C\otimes P_C)+\gamma\Gamma^2(F^2)-\delta F\otimes F+\tfrac{1}{3}\nu K\otimes K+\lambda(P_{|F|}\otimes P_{|F|})M$$

$$+\eta(F+I)\otimes(F+I)(S\otimes I+I\otimes S)$$

where $\lambda=29.33$ and $\eta=4.5$.

The operator M is independent of color and has the form $M=I\otimes I\otimes\hat{M}$ where I is the identity on the color space and $\hat{M}$ is a diagonal 288×288 matrix. Instead of writing out the 288 nonzero entries of $\hat{M}$, we use the following condensed notation. Let $q_i\uparrow$ represent a quark with flavor f_i, $i=1,\ldots,6$, and spin up. We have similar definitions for $q_i\downarrow$, $\bar{q}_i\uparrow$, and $\bar{q}_i\downarrow$. Define $\hat{M}$ by

$$\hat{M}q_i\uparrow\otimes\bar{q}_j\downarrow=\lambda(i,j)q_i\uparrow\otimes\bar{q}_j\downarrow$$

$$\hat{M}q_i\downarrow\otimes\bar{q}_j\uparrow=\lambda(i,j)q_i\downarrow\otimes\bar{q}_j\uparrow$$

$$\hat{M}\bar{q}_i\uparrow\otimes q_j\downarrow=\lambda(i,j)\bar{q}_i\uparrow\otimes q_j\downarrow$$

$$\hat{M}\bar{q}_i\downarrow\otimes q_j\uparrow=\lambda(i,j)\bar{q}_i\downarrow\otimes q_j\uparrow$$

and

$$\hat{M}q_i\uparrow\otimes\bar{q}_j\uparrow=\lambda'(i,j)q_i\uparrow\otimes\bar{q}_j\uparrow$$

$$\hat{M}q_i\downarrow\otimes\bar{q}_j\downarrow=\lambda'(i,j)q_i\downarrow\otimes\bar{q}_j\downarrow$$

$$\hat{M}\bar{q}_i\uparrow\otimes q_j\uparrow=\lambda'(i,j)\bar{q}_i\uparrow\otimes q_j\uparrow$$

$$\hat{M}\bar{q}_i\downarrow\otimes q_j\downarrow=\lambda'(i,j)\bar{q}_i\downarrow\otimes q_j\downarrow$$

where $\lambda(i, j)$ and $\lambda'(i, j)$ are the following matrices:

$$[\lambda(i,j)] = \begin{bmatrix} 1 & 1 & 2 & 9 & \frac{5}{8} & \frac{9}{8} \\ 1 & 1 & 2 & 9 & \frac{5}{8} & \frac{9}{8} \\ 2 & 2 & 2 & 4 & \frac{5}{8} & \frac{9}{8} \\ 9 & 9 & 4 & 4 & \frac{5}{8} & \frac{9}{8} \\ \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{9}{8} \\ \frac{9}{8} & \frac{9}{8} & \frac{9}{8} & \frac{9}{8} & \frac{9}{8} & \frac{9}{8} \end{bmatrix}$$

$$[\lambda'(i,j)] = \begin{bmatrix} 2 & 2 & 0 & 4 & \frac{9}{16} & \frac{17}{16} \\ 2 & 2 & 0 & 4 & \frac{9}{16} & \frac{17}{16} \\ 0 & 0 & 0 & 2 & \frac{9}{16} & \frac{17}{16} \\ 4 & 4 & 2 & 3 & \frac{9}{16} & \frac{17}{16} \\ \frac{9}{16} & \frac{9}{16} & \frac{9}{16} & \frac{9}{16} & \frac{9}{16} & \frac{17}{16} \\ \frac{17}{16} & \frac{17}{16} & \frac{17}{16} & \frac{17}{16} & \frac{17}{16} & \frac{17}{16} \end{bmatrix}$$

The matrix elements can be defined in terms of the numbers $f_1 = 1$, $f_i = 2^{i-2}$, $i = 2, \ldots, 6$, which are essentially the moduli of the flavors, as follows:

$$\lambda(i,j) = \begin{cases} \left[\max(f_i, f_j) + 2\right]/16, & \text{if } i \text{ or } j \geqslant 5 \\ \max(f_i, f_j) + 5, & \text{if } i = 4 \text{ and } j \leqslant 2, \text{ or } j = 4 \text{ and } i \leqslant 2 \\ \max(f_i, f_j), & \text{otherwise} \end{cases}$$

$$\lambda'(i,j) = \begin{cases} \left[\max(f_i, f_j) + 1\right]/16, & \text{if } i \text{ or } j \geqslant 5 \\ i + j - 5, & \text{if } 3 \leqslant i \text{ and } j \leqslant 4 \\ \lambda(i,j) + 1, & \text{if } \lambda(i,j) = 1 \\ \lambda(i,j) - 2, & \text{if } \lambda(i,j) = 2 \\ \max(f_i, f_j), & \text{otherwise} \end{cases}$$

In the following, we list the quark composition followed by the mass expectation for the mesons. These values fall within 1% of the experimental values.

$$\pi^0 = (2)^{-1/2}(u_r\uparrow \otimes \bar{u}_r\downarrow + \bar{u}_r\downarrow \otimes u_r\uparrow)$$

$$\langle H^{(2)}\pi^0, \pi^0\rangle = 2\gamma - \delta + \left(1\tfrac{1}{64}\right)^2\lambda + \tfrac{1}{3}\left(\tfrac{2}{3}\right)\left(-\tfrac{2}{3}\right)\nu = 135$$

$$\pi^- = (2)^{-1/2}(d_r\uparrow \otimes \bar{u}_r\downarrow + \bar{u}_r\downarrow \otimes d_r\uparrow)$$

$$\langle H^{(2)}\pi^-, \pi^-\rangle = \left[\left(1\tfrac{1}{32}\right)^2 + 1\right]\gamma - \left(1\tfrac{1}{32}\right)\delta + \left(1\tfrac{1}{64}\right)^2\lambda + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)\nu = 140$$

$$\pi^+ = \bar{\pi}^-$$

$$\langle H^{(2)}\pi^+, \pi^+\rangle = 140$$

$$K^+ = (2)^{-1/2}(u_r\uparrow \otimes \bar{s}_r\downarrow + \bar{s}_r\downarrow \otimes u_r\uparrow)$$

$$\langle H^{(2)}K^+, K^+\rangle = 5\gamma + 2\delta + 6\left(1\tfrac{1}{64}\right)\lambda + \tfrac{1}{3}\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)\nu = 496$$

$$K^- = \bar{K}^+$$

$$\langle H^{(2)}K^-, K^-\rangle = 496$$

$$K^0 = (2)^{-1/2}(d_r\uparrow \otimes \bar{s}_r\downarrow + \bar{s}_r\downarrow \otimes d_r\uparrow)$$

$$\langle H^{(2)}K^0, K^0\rangle = \left[\left(1\tfrac{1}{32}\right)^2 + 4\right]\gamma + 2\left(1\tfrac{1}{32}\right)\delta + 6\left(1\tfrac{1}{64}\right)\lambda + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\nu = 500$$

$$\eta' = (2)^{-1/2}(s_r\uparrow \otimes \bar{s}_r\downarrow + \bar{s}_r\downarrow \otimes s_r\uparrow)$$

$$\langle H^{(2)}\eta', \eta'\rangle = 8\gamma - 4\delta + 18\lambda + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\nu = 953$$

$$X = (2)^{-1/2}(c_r\uparrow \otimes \bar{c}_r\downarrow + \bar{c}_r\downarrow \otimes c_r\uparrow)$$

$$\langle H^{(2)}X, X\rangle = 32\gamma - 16\delta + 36\lambda + \tfrac{1}{3}\left(\tfrac{2}{3}\right)\left(-\tfrac{2}{3}\right)\nu = 2750$$

$$\rho^0 = (6)^{-1/2}(u_r\uparrow \otimes \bar{u}_r\uparrow + u_y\uparrow \otimes \bar{u}_y\uparrow + u_b\uparrow \otimes \bar{u}_b\uparrow - \bar{u}_r\uparrow \otimes u_r\uparrow - \bar{u}_y\uparrow \otimes u_y\uparrow - \bar{u}_b\uparrow \otimes u_b\uparrow)$$

$$\langle H^{(2)}\rho^0, \rho^0\rangle = 6\left|1 + \sqrt{3}/2 + \tfrac{1}{2}i\right|^2\beta + 2\gamma - \delta + 2\left(1\tfrac{1}{64}\right)^2\lambda + 4\eta + \tfrac{1}{3}\left(\tfrac{2}{3}\right)\left(-\tfrac{2}{3}\right)\nu = 776$$

$\rho^+ = u\bar{d}$ (color symmetrization as in ρ^0, similarly for remaining mesons)

$$\langle H^{(2)}\rho^+, \rho^+\rangle = 6\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + \left[\left(1\tfrac{1}{32}\right)^2+1\right]\gamma - \left(1\tfrac{1}{32}\right)\delta + 2\left(1\tfrac{1}{64}\right)^2\lambda$$

$$+2\left(2\tfrac{1}{32}\right)\eta + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(-\tfrac{2}{3}\right)\nu = 781$$

$$\rho^- = \bar{\rho}^+$$

$$\langle H^{(2)}\rho^-, \rho^-\rangle = 781$$

$$\omega^0 = d\bar{d}$$

$$\langle H^{(2)}\omega^0, \omega^0\rangle = 6\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + 2\left(1\tfrac{1}{32}\right)^2\gamma - \left(1\tfrac{1}{32}\right)^2\delta + 2\left(1\tfrac{1}{64}\right)^2\lambda$$

$$+\left(2\tfrac{1}{32}\right)^2\eta + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\nu = 785$$

$$K^{*+} = u\bar{s}$$

$$\langle H^{(2)}K^{*+}, K^{*+}\rangle = 6\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + 5\gamma + 2\delta - 2\eta + \tfrac{1}{3}\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)\nu = 901$$

$$K^{*0} = d\bar{s}$$

$$\langle H^{(2)}K^{*0}, K^{*0}\rangle = 6\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + \left[\left(1\tfrac{1}{32}\right)^2+4\right]\gamma + 2\left(1\tfrac{1}{32}\right)\delta - \left(2\tfrac{1}{32}\right)\eta$$

$$+\tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\nu = 905$$

$$\phi^0 = s\bar{s}$$

$$\langle H^{(2)}\phi^0, \phi^0\rangle = 6\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + 8\gamma - 4\delta$$

$$+\eta + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\nu = 1020$$

$$\psi/J = c\bar{c}$$

$$\langle H^{(2)}\psi/J, \psi/J\rangle = 6\left|1+\sqrt{3}/2+\tfrac{1}{2}i\right|^2\beta + 32\gamma - 16\delta$$

$$+27\lambda + 9\eta + \tfrac{1}{3}\left(\tfrac{2}{3}\right)\left(-\tfrac{2}{3}\right)\nu = 3120$$

It appears as though η^0 is an exception. If we define $\hat{\eta} = (2)^{-1/2}(d_r\uparrow\otimes \bar{d}_r\downarrow + \bar{d}_r\downarrow\otimes d_r\uparrow)$, then

$$\langle H^{(2)}\hat{\eta}, \hat{\eta}\rangle = 2\left(1\tfrac{1}{32}\right)^2\gamma - \left(1\tfrac{1}{32}\right)^2\delta + \left(1\tfrac{1}{64}\right)^2\lambda + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\nu = 143$$

but this mass does not seem to have been observed. We conjecture that

$\eta^0 = (2)^{-1/2}(\eta' + \hat{\eta})$. If this is the case, then

$$\langle H^{(2)}\eta^0, \eta^0\rangle = \tfrac{1}{2}(\langle H^{(2)}\eta', \eta'\rangle + \langle H^{(2)}\hat{\eta}, \hat{\eta}\rangle) = 548$$

Finally, we summarize our calculations for some recently discovered mesons which still have fairly large experimental errors:

$$D^0 = u\bar{c}$$
$$\langle H^{(2)}D^0, D^0\rangle = 1849$$
$$\langle H^{(2)}D^{*0}, D^{*0}\rangle = 1967$$
$$D^- = d\bar{c}$$
$$\langle H^{(2)}D^-, D^-\rangle = 1855$$
$$\langle H^{(2)}D^{*-}, D^{*-}\rangle = 1974$$
$$F^- = s\bar{c}$$
$$\langle H^{(2)}F^-, F^-\rangle = 2139$$
$$\langle H^{(2)}F^{*-}, F^{*-}\rangle = 2217$$
$$Y^0 = b\bar{b}$$
$$\langle H^{(2)}Y^0, Y^0\rangle = 9423$$
$$\langle H^{(2)}Y^{*0}, Y^{*0}\rangle = 10116$$

One might object that with the number of constants included in the above Hamiltonians, one can predict anything. This is certainly a valid argument. However, except for the eight basic constants, the others follow a fairly regular pattern, and the number of predicted masses far exceeds the number of constants.

REFERENCES

Close, F. E. (1979). *An Introduction to Quarks and Partons*. Academic Press, New York.

Gudder, S. (1982). A logical explanation for quarks, *Foundations in Physics*, **12**, 419.

Gudder, S., and Naroditsky, V. (1981). Finite dimensional quantum mechanics, *International Journal of Theoretical Physics*, **20**, 619.